PHYS5150 — PLASMA PHYSICS

LECTURE 9 - TRAPPING IN DIPOLAR FIELDS

Sascha Kempf[*]

G2B41, University of Colorado, Boulder

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1 TRAPPING IN DIPOLAR MAGNETIC FIELDS

1.1 *Dipole field*

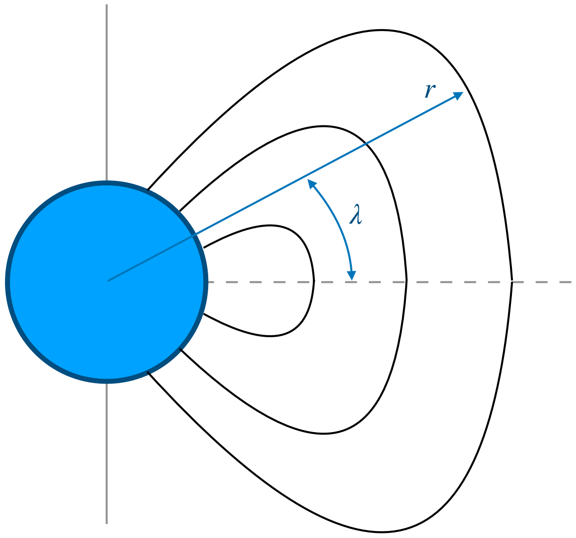
The magnetic dipole field has a field strength of

$$\mathbf{B}_{dipole} = \frac{\mu_0}{4\pi r^2}\frac{M}{r}\left(-2\sin\lambda\hat{\mathbf{e}}_r + \cos\lambda\hat{\mathbf{e}}_\lambda\right) \tag{1}$$

and

$$B_{dipole} = \frac{\mu_0}{4\pi r^2}\frac{M}{r}\left(1+3\sin^2\lambda\right)^{1/2}, \tag{2}$$

where λ is the *magnetic latitude* and M is the magnetic moment. As an example, $M = 8.05\cdot 10^{22}\,\mathrm{A\,m^2}$ is the magnetic moment of the Earth' dipolar field.



*sascha.kempf@colorado.edu

We need an equation for a magnetic field line, i.e. an expression for $B(\lambda)$ along a *line of force*. If $\mathrm{d}\mathbf{s} = (\mathrm{d}r, \lambda\,\mathrm{d}\lambda)$ is an arc element, then a line of force satisfies

$$\mathrm{d}\mathbf{s} \times \mathbf{B} \stackrel{!}{=} 0,$$

from which follows that

$$\frac{\mathrm{d}r}{B_r} = \frac{r\,\mathrm{d}\lambda}{B_\lambda}.$$

Using Eq. (1)

$$\frac{\mathrm{d}r}{r} = -\frac{2\sin\lambda\,\mathrm{d}\lambda}{\cos\lambda} = \frac{2\mathrm{d}(\cos\lambda)}{\cos\lambda},$$

and after integration

$$r(\lambda) = r(0)\cos^2\lambda. \tag{3}$$

r is usually expressed as multiple $L = r(0)/R_p$ of the planet's equatorial radius R_p. Substituting Eq. (3) into Eq. (2) and introducing the equatorial field strength at the planet's surface

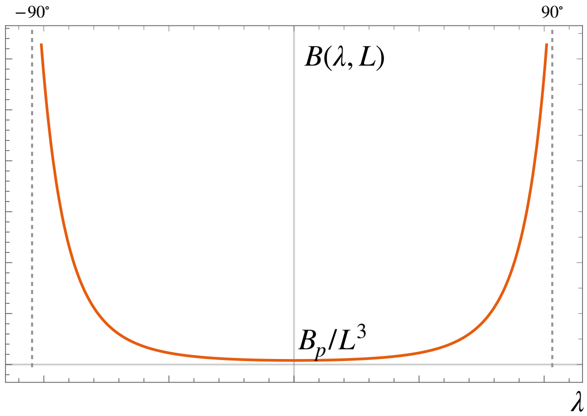
$$B_p = \frac{\mu_0 M_p}{4\pi R_p^3}$$

gives the standard form for a planetary magnetic dipolar field

$$B(\lambda, L) = \frac{B_p}{L^3}\frac{\left(1+3\sin^2\lambda\right)^{1/2}}{\cos^6\lambda} \tag{4}$$

$$\cos^2\lambda_p = L^{-1}, \tag{5}$$

where λ_p is the latitude at which a field line of a given L dissects the planet's surface.



As apparent from the figure above, a planetary dipolar field is in fact a magnetic mirror and is thus able to trap plasma particles.

1.2 *Bouncing in a dipole field*

Let' introduce the so-called *pitch angle* between the particle's velocity vector and the local magnetic field

$$\alpha = \tan^{-1}\frac{v_\perp}{v_\parallel}.$$

Because of $v = v_\perp \sin\alpha$,

$$\mu = \frac{T_\perp}{B} = \frac{mv^2}{2B}\sin^2\alpha = \frac{T}{B}\sin^2\alpha,$$

and $\dot{T} = 0$, the pitch angle depends only on the magnetic field strength B. Hence,

$$\frac{\sin^2\alpha_1}{B_1} = \frac{\sin^2\alpha_2}{B_2},$$

or

$$\frac{\sin^2\alpha_1}{\sin^2\alpha_2} = \frac{B_1}{B_2}.$$

At the mirror points is $T_\parallel = 0$ and therefore $\alpha = 90°$, and thus

$$\boxed{\sin\alpha = \left(\frac{B}{B_{mp}}\right)^{1/2}.} \tag{6}$$

This means that the pitch angle at a given location depends on the ratio between the local field strength B and the field strength at the mirror point B_{mp}.

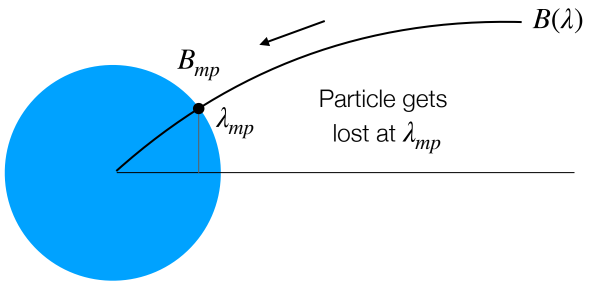
It is useful to define the pitch angle α_{eq} at the equatorial plane

$$\sin^2\alpha_{eq} = \frac{B_{eq}}{B_{mp}} = \frac{\cos^6\lambda_{mp}}{(1+3\sin^2\lambda_{mp})^{1/2}},$$

where λ_{mp} is the latitude of the mirror point. α_{eq} depends only on λ_{mp}, but not on L. Small equatorial pitch angles imply large values of $v_\parallel$.

1.3 *Loss cone*

Particles may for instance collide with neutrals of the planet's atmosphere and get lost. Let's assume that this happens at zero altitude.

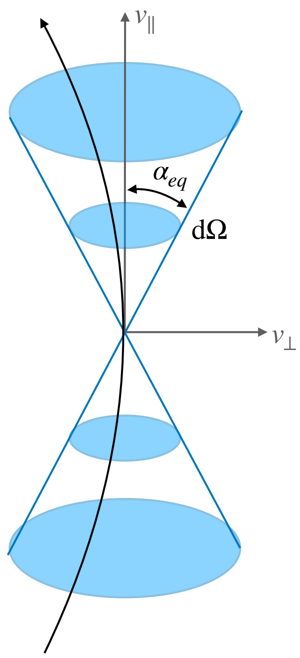


In this case we can use the equations for the equatorial pitch angle introduced before, i.e.

$$\sin^2 \alpha_{eq} = \frac{B_{eq}}{B_{mp}} = \frac{\cos^6 \lambda_{mp}}{(1 + 3\sin^2 \lambda_{mp})^{1/2}}.$$

Particles with pitch angles $\alpha < \alpha_{eq}$ will be lost to the atmosphere. The same applies to particles with $\alpha > 180^\circ - \alpha_{eq}$.

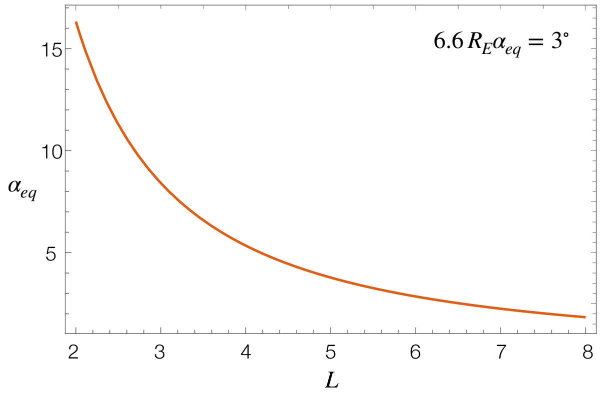
The geometric interpretation of this result is this of a *loss cone*:



Expressing λ_{mp} by its corresponding L value gives

$$\boxed{\sin^2 \alpha_{eq} = \left(4L^6 - 3L^5\right)^{-1/2}.} \tag{7}$$

Note that the loss cone does not depend on the particle charge, mass, and energy, but only on the filed line curvature.



Remains the question under what circumstances trapping can actually happen. We already found that

$$T = \mu B_{mp} = \frac{1}{2} m v_\parallel^2 + \mu B.$$

Particles will get trapped if their kinetic energy

$$T < \mu B_{mp}.$$

Now,

$$\mu B_m = \frac{T_\perp}{B} B_{mp} = \frac{m}{2} v_{\perp,eq}^2 \frac{B_{mp}}{B_{eq}} > \frac{m}{2} \left\{ v_{\perp,eq}^2 + v_{\parallel,eq}^2 \right\},$$

and

$$v_{\perp,eq}^2 \left(\frac{B_{mp}}{B_{eq}} - 1 \right) \geq v_{\parallel,eq}^2.$$

After substituting for the mirror ratio $R = B_{mp}/B_{eq}$ we get

$$v_{\perp,eq}^2 (R-1) \geq v_{\parallel,eq}^2,$$

or

$$\boxed{\left| \frac{v_{\perp,eq}}{v_{\parallel,eq}} \right| = \left| \tan \alpha_{eq} \right| \geq \frac{1}{\sqrt{R-1}}.}$$

1.4 *Bounce period*

The duration τ_b for a single bounce is

$$\tau_b = 4 \int_0^{\lambda_{mp}} \frac{\mathrm{d}s}{v_\parallel} = 4 \int_0^{\lambda_{mp}} \frac{\mathrm{d}s}{\mathrm{d}\lambda} \frac{\mathrm{d}\lambda}{v_\parallel}.$$

Using that guiding center velocity along the field line is

$$v_{\parallel} = v\left(1 - \frac{B}{B_{eq}}\sin^2\alpha_{eq}\right)^{1/2}$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(r_{eq}\cos^2\lambda\right) = r_{eq}\cos\lambda\left(1+3\sin^2\lambda\right)^{1/2},$$

we obtain

$$\tau_b = 4\frac{r_{eq}}{v}\int_0^{\lambda_{mp}}\cos\lambda(1+3\sin^2\lambda)^{1/2}\left[1-\sin^2\alpha_{eq}\frac{(1+3\sin^2\lambda)^{1/2}}{\cos^6\lambda}\right]^{-1/2}\mathrm{d}\lambda.$$

The integral cannot be solved analytical, but is reasonably well approximated by

$$\tau_b \approx \frac{L\cdot R_p}{(T/m)^{1/2}}\left(3.7 - 1.6\sin\alpha_{eq}\right).$$

The bouncing period exhibits only a weak dependence on the equatorial pitch angle. This is resulting from the fact that small α_{eq} correspond to large values of $v_{\parallel}$, and vice versa.

1.5 *Drifts*

Finally let us consider the drifts in an dipolar field. In a cylindrically symmetric field configuration is

$$-\nabla B = \left(\frac{B}{R_c^2}\right)\hat{\mathbf{R}}_c.$$

This allows to merge v_R and v_G into a single expression. The total magnetic drift is then

$$\boxed{\mathbf{v}_B = \mathbf{v}_R + \mathbf{v}_G = \left(v_{\parallel}^2 + \frac{1}{2}v_{\perp}^2\right)\frac{\mathbf{B}\times\nabla B}{\omega_c B^2}.}$$

The angular drift per bounce cycle is then

$$\Delta\Psi = 4\int_0^{\lambda_{mp}}\frac{\mathbf{v}_B}{r\cos\lambda}\frac{\mathrm{d}s}{v_{\parallel}},$$

which allows us to find the drift time scale

$$\langle\ \tau_d\ \rangle \approx \frac{\pi q B_p R_p^2}{3LT}\left(0.35 + 0.15\sin\alpha_{eq}\right)^{-1}$$

as well as the average drift velocity of the bouncing particles

$$\langle\ v_d\ \rangle \approx \frac{6L^2T}{qB_pR_p}\left(0.35+0.15\sin\alpha_{eq}\right).$$

Note that $\langle\, v_d \,\rangle$ does not depend on the particle mass. This means that ions and electrons drift with the same $\langle\, v_d \,\rangle$ in opposite directions. Also, because $\langle\, v_d \,\rangle$ scales as L^2, the drift velocity at more distant L shells is actually faster.